Fractional calculus models for the earth surface: Success to date, challenges, and opportunities

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Stochastic transport and emergent scaling in Earth-surface processes (STRESS 2) workshop

Tahoe Environmental Research Center , Incline Village NV

4 – 6 November 2009

Partially supported by NSF grants DMS-0803360 and EAR-0823965.

Abstract

Fractional calculus and heavy tailed stochastic processes have already proven useful to model anomalous diffusion/dispersion on the earth surface. Many open problems remain, including experimental efforts to verify power law statistics, incorporation of waiting times between particle movements, development of fractional models to explain multifractal behavior, tempered or truncated power laws, and other alternative models between Gaussian and power law.

Collaborators

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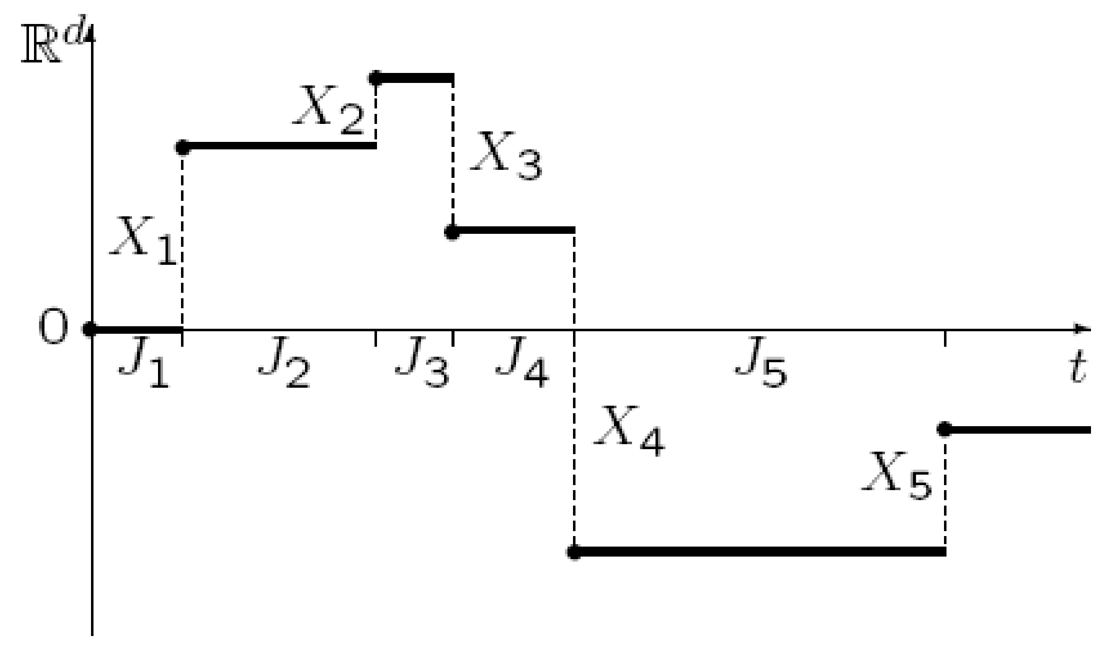
Continuous time random walk



Particle location: $S(n) = X_1 + \cdots + X_n$

Jump time: $T(n) = J_1 + \cdots + J_n$

Exner equation and CTRW

Exner equation: $$(1-\lambda_p)\frac{\partial\eta}{\partial t}=D_b(x,t)-E_b(x,t)$$

Here η=elevation, λ_p=porosity, D_b=deposition rate, E_b=entrainment rate.

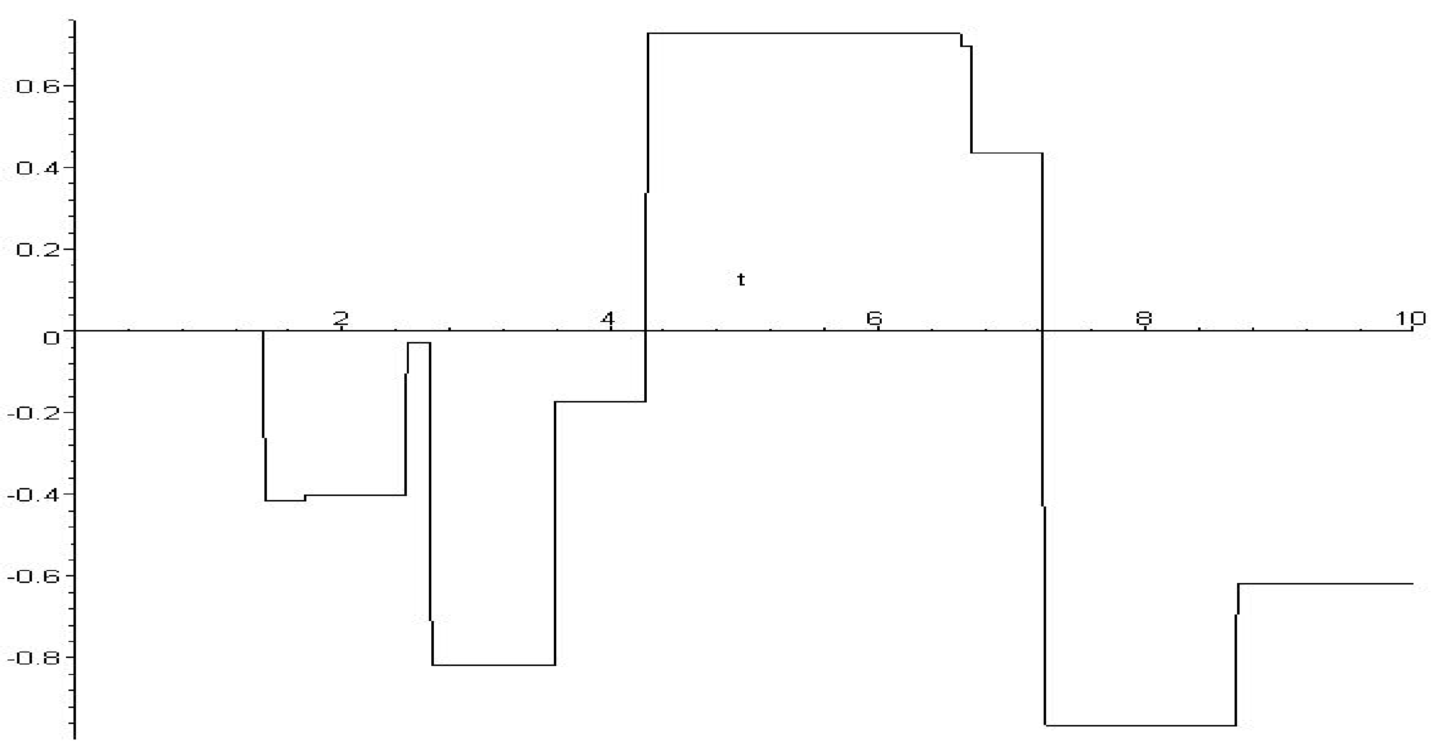
An active layer formulation leads to the probabilistic Exner equation:

$$(1-\lambda_p)\frac{L_a}{E_b}\frac{\partial f_a(x,t)}{\partial t}=\int_0^\infty f_a(x-r,t)\,f_s(r)dr-f_a(x,t)$$

Here L_a=active layer thickness, f_a=fraction of particles in the active layer, and f_s=step length pdf for particle jump X. See Ganti et al. [5].

This equation governs the pdf of a CTRW with exponential waiting times having mean $\lambda=(1-\lambda_p)\,L_a/E_b$.

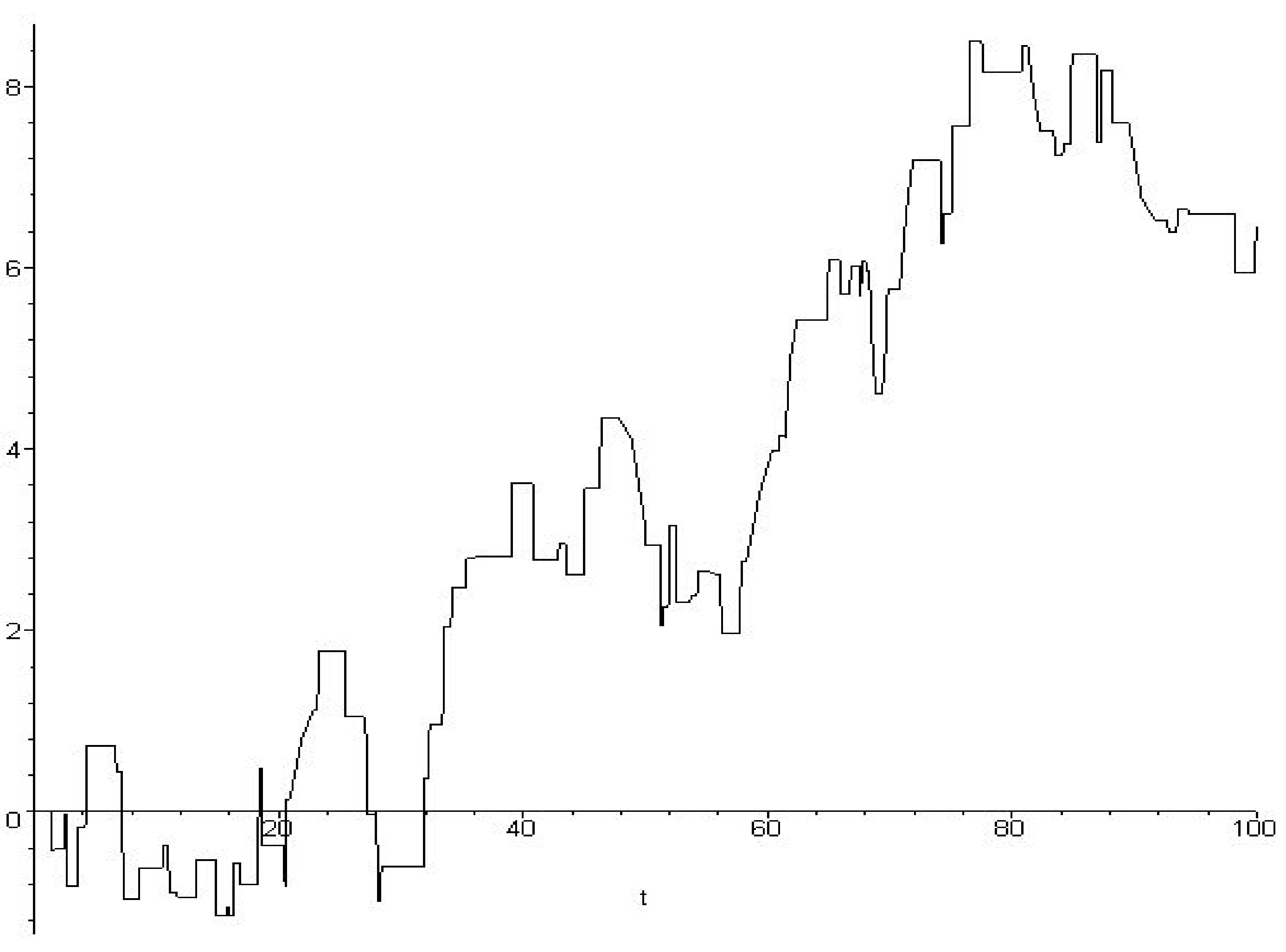
CTRW simulation



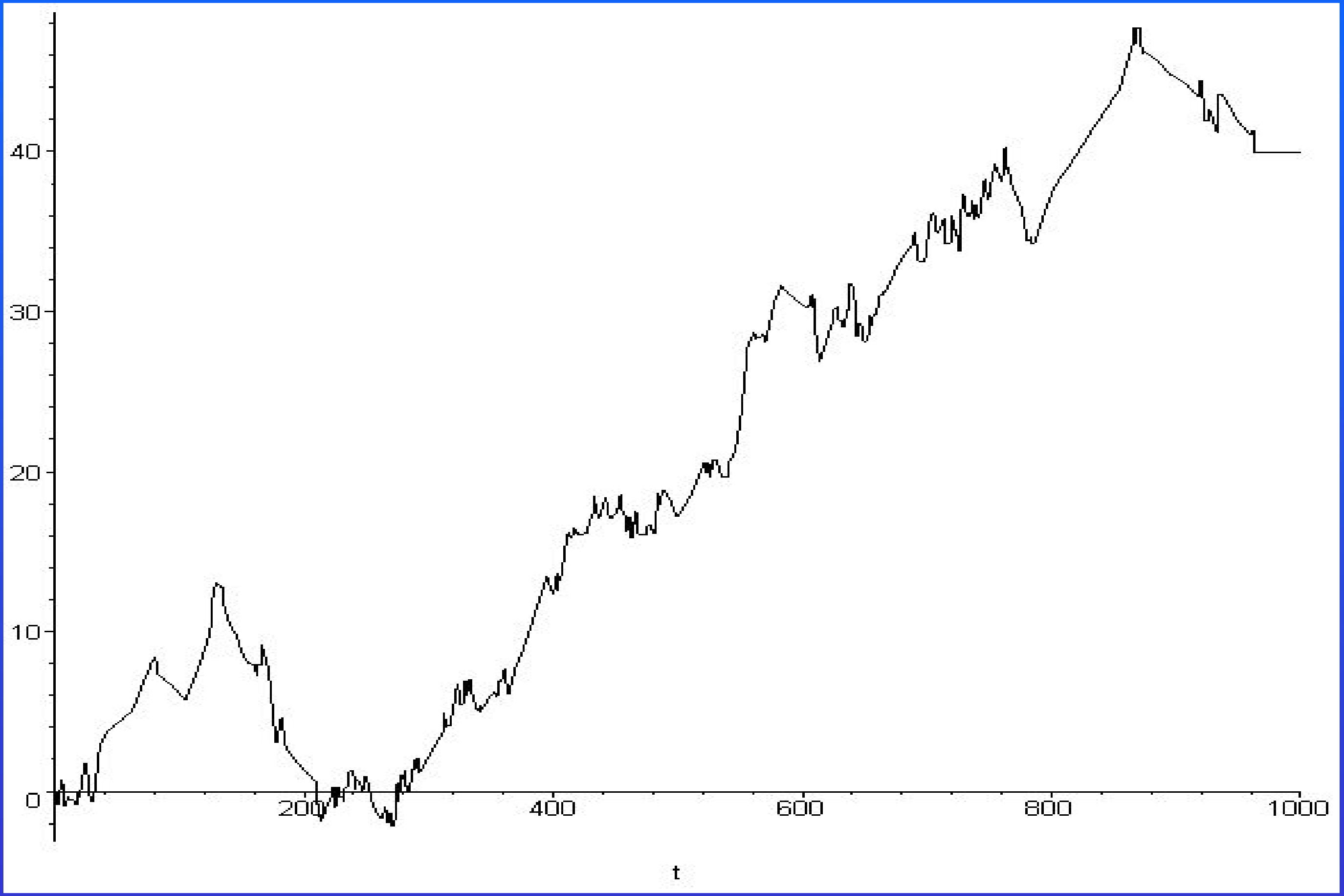
Jumps are uniform on [-1,1]

Waiting times are exponential with mean 1.

More jumps



Long time limit: Brownian motion



Long time limit process is *statistically* predictable [11].

Probabilistic Exner equation and ADE

Ganti et al. [5] compute the long time limit ($\lambda_p = 0$):

$$\frac{L_a}{E_b}\frac{\partial f_a}{\partial t} = -v\frac{\partial f_a}{\partial x} + D_d\frac{\partial^2 f_a}{\partial x^2}$$

This *Advection Dispersion Equation* governs a Brownian motion.

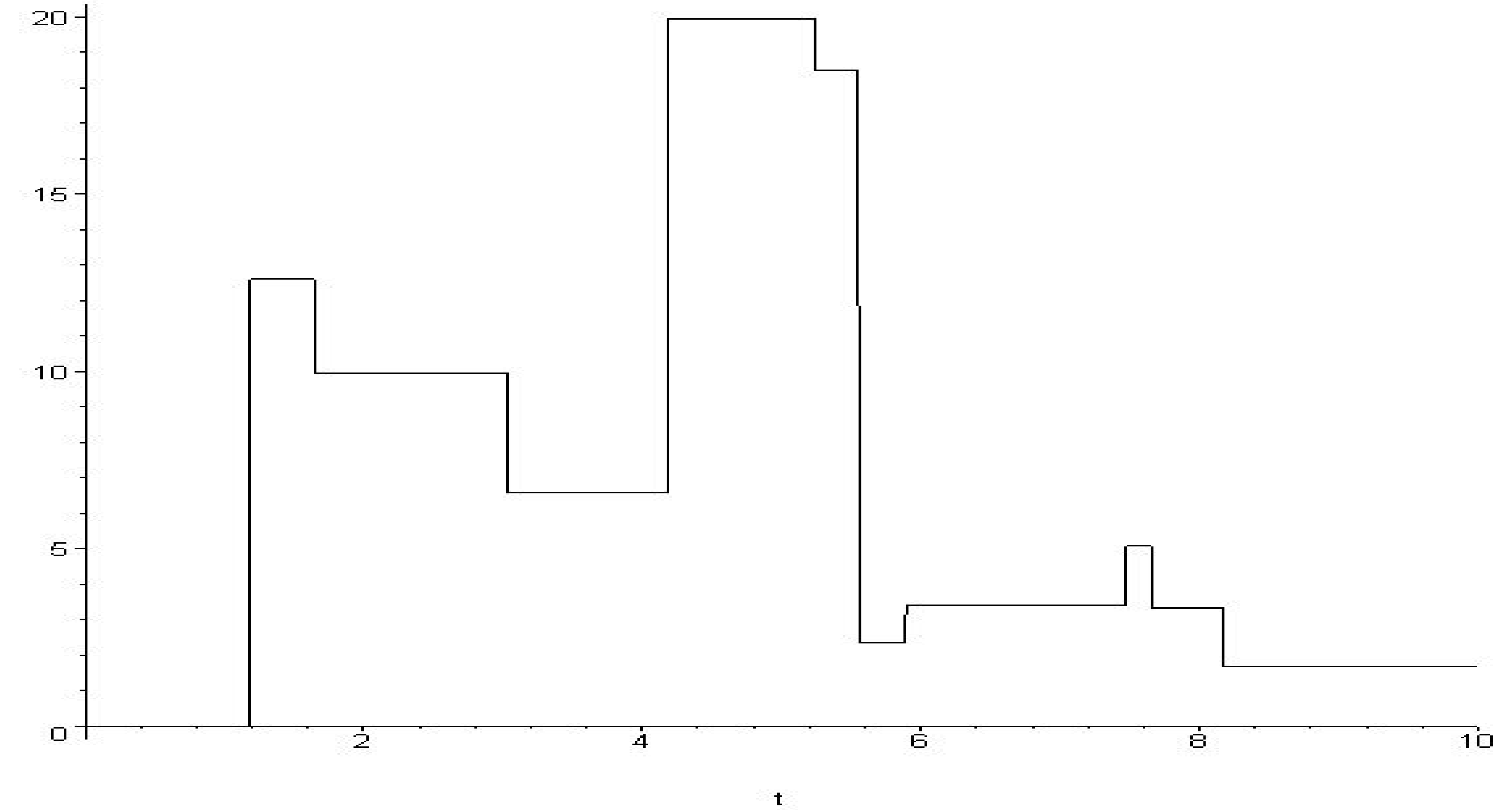
The ADE is valid if the step length pdf f_s has finite mean and variance.

Infinite variance jumps lead to the fractional ADE of Benson et al. [2]:

$$\frac{L_a}{E_b}\frac{\partial f_a}{\partial t} = -v\frac{\partial f_a}{\partial x} + D_d\frac{\partial^\alpha f_a}{\partial x^\alpha}$$

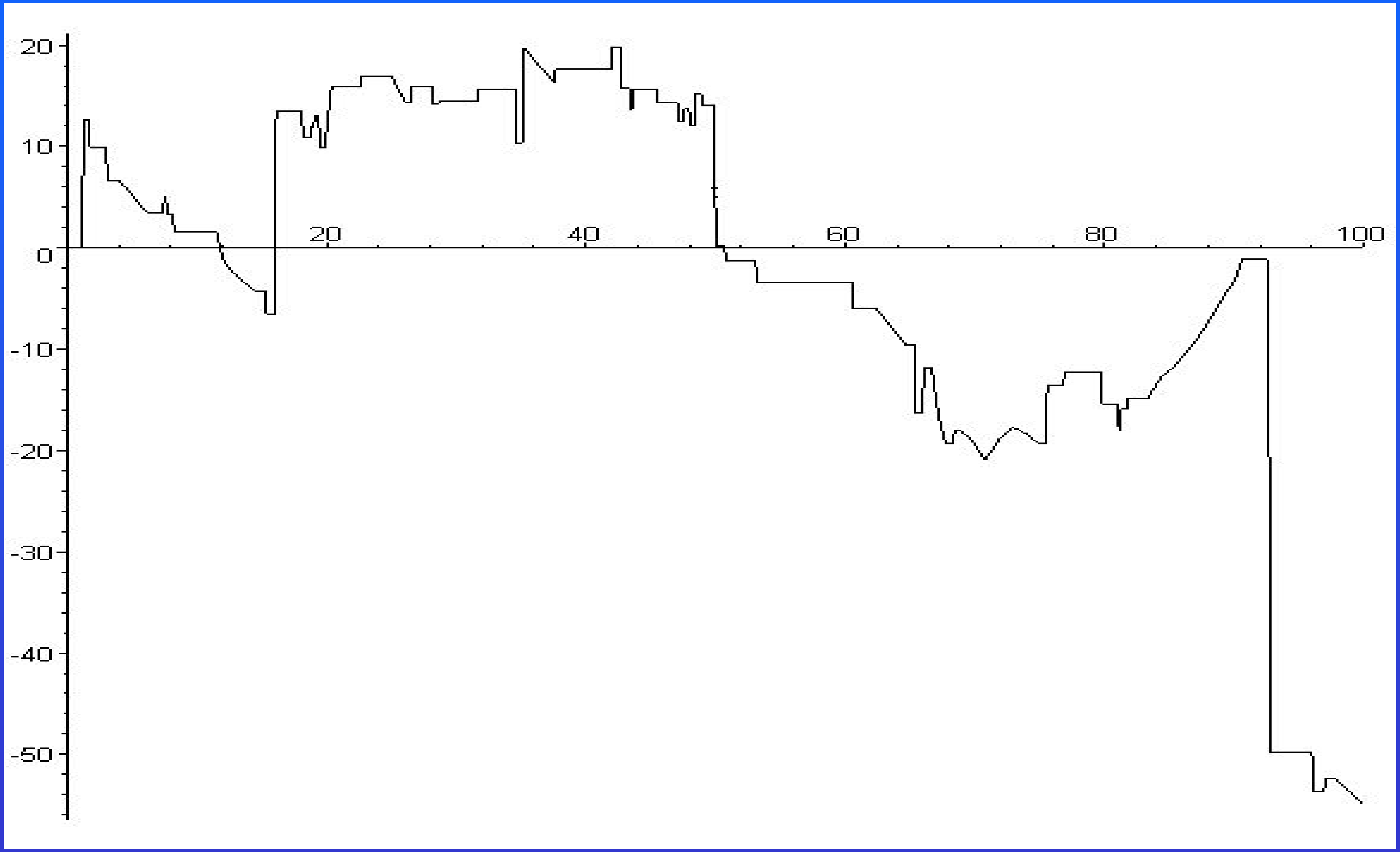
Valid for power law jumps: $P(X > x) \approx Cx^{-\alpha} \quad 0 < \alpha < 2$

CTRW simulation with long particle jumps

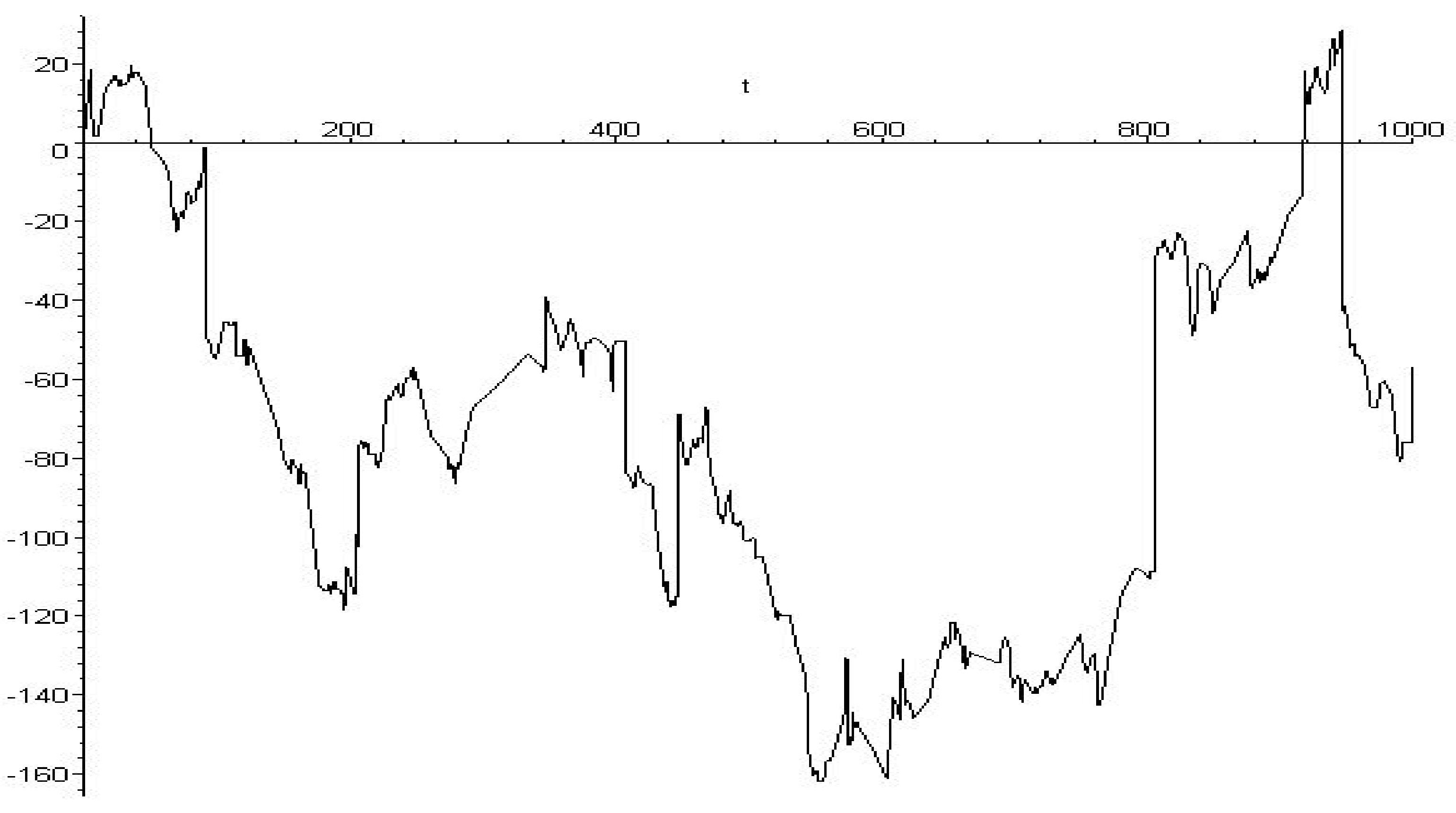


$$P(|X| > x) \approx Cx^{-\alpha}, \quad C = 1, \quad \alpha = 1.5$$

Longer time scale



Long time limit: Stable Lévy motion



Limit retains large particle jumps, see Benson et al. [2].

CTRW with long waiting times

The CTRW with infinite mean waiting times has a long time limit governed by a time-fractional ADE [8]:

$$\frac{\partial^\gamma C(x,t)}{\partial t^\gamma} = -v\frac{\partial C(x,t)}{\partial x} + \mathcal{D}\frac{\partial^\alpha C(x,t)}{\partial x^\alpha}$$

Valid for power law waiting times: $P(J > t) \approx Cx^{-\gamma} \quad 0 < \gamma < 1$

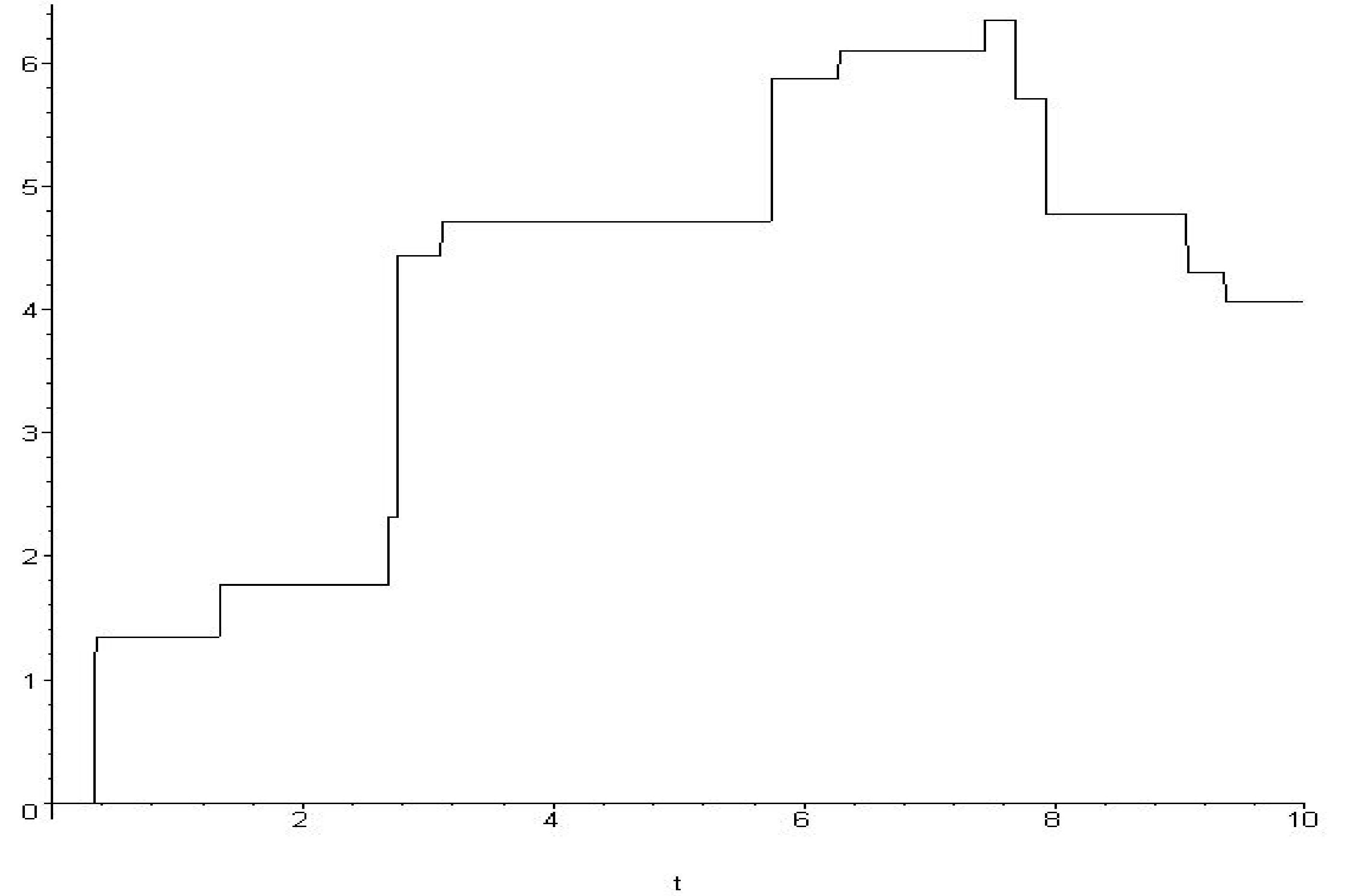
For finite variance jumps, take $\alpha=2$ (classical in space)

For infinite variance jumps, take $\alpha<2$ (fractional in space)

Schumer et al. [15] apply to particle transport.

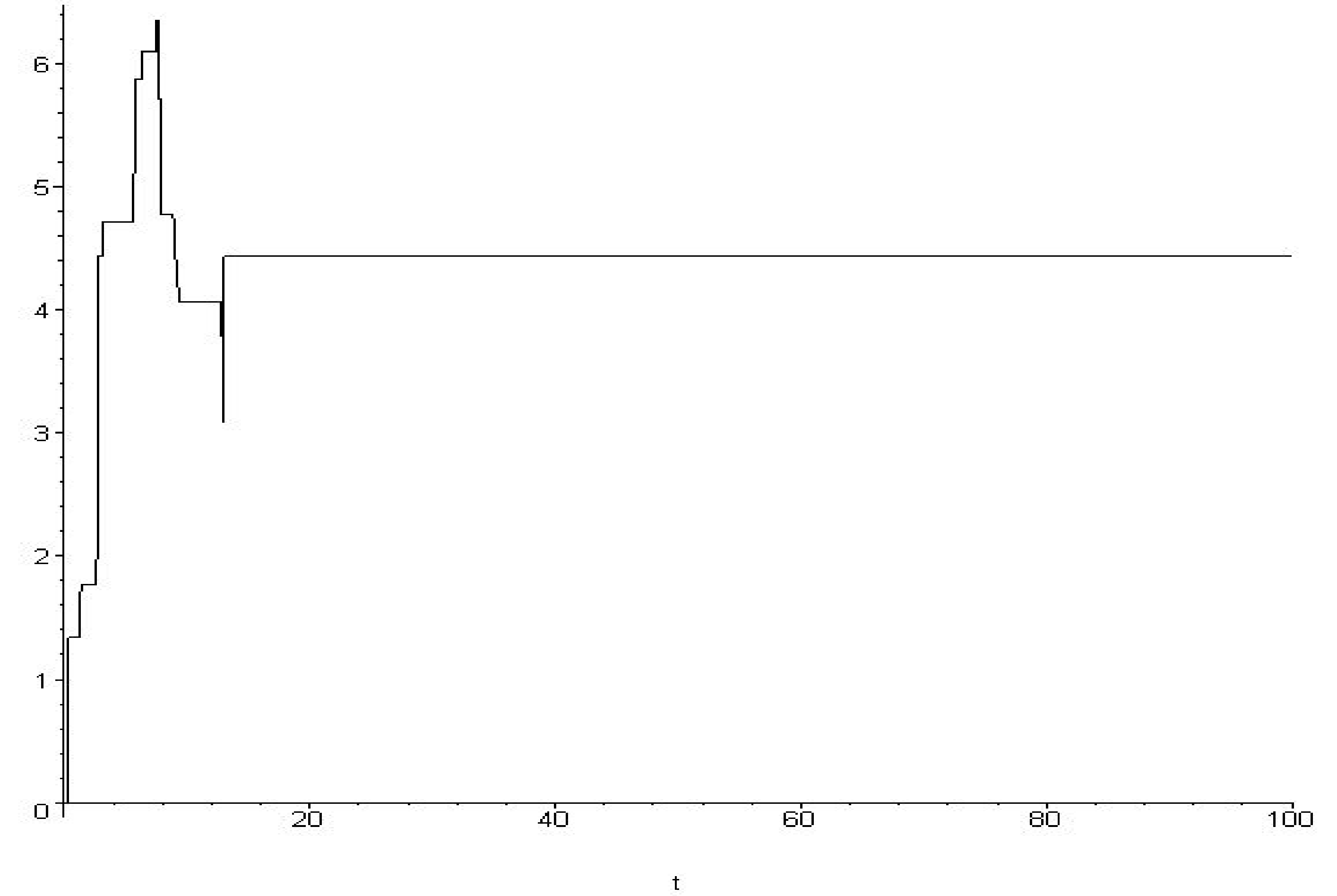
Schumer and Jerolmack [14] apply to geological records.

CTRW with long waiting times

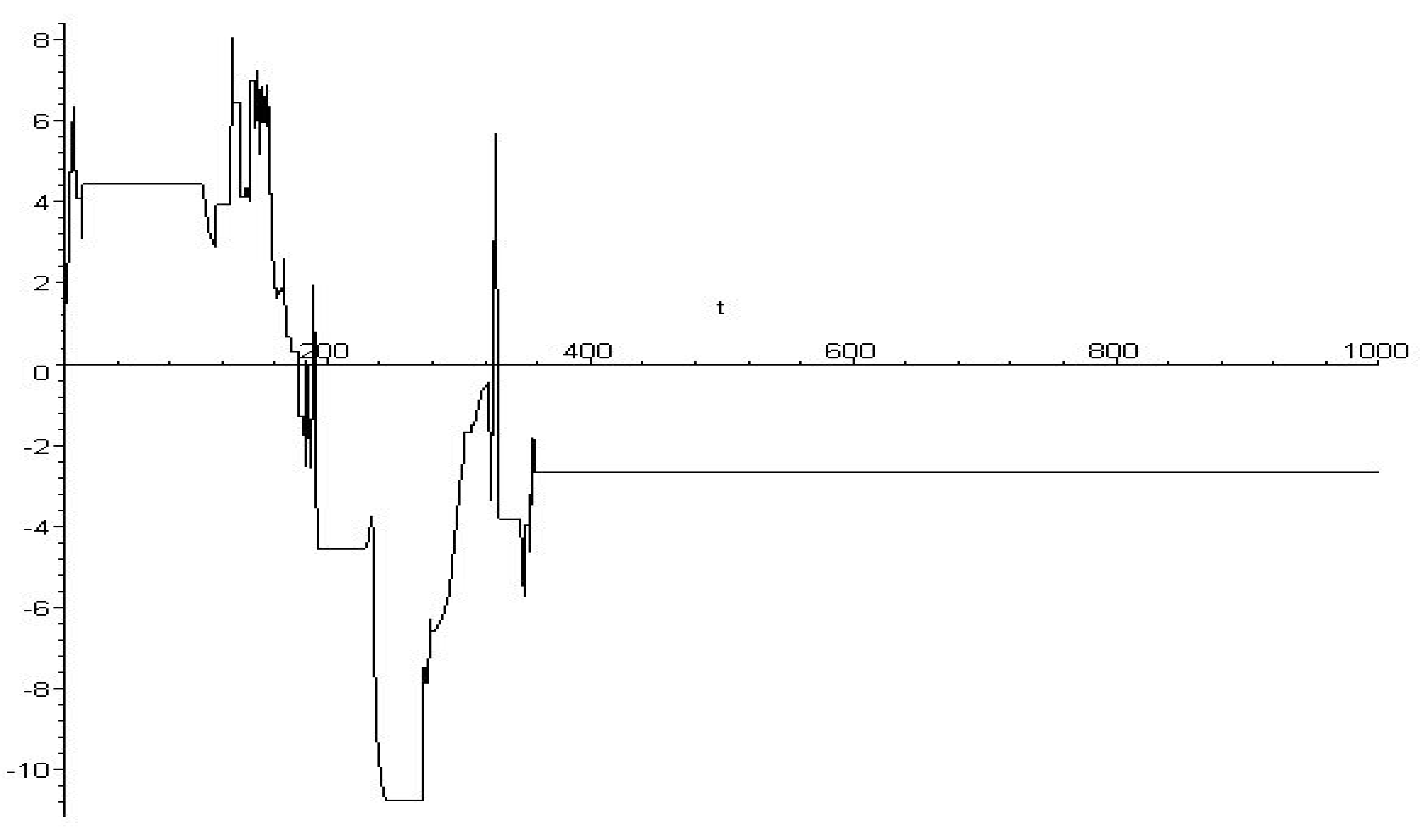


$$P(J > t) \approx Ct^{-\beta}, \quad C = 0.3, \quad \beta = 0.9$$

More jumps



Long time limit: Subordinated motion



Flat portions are particle resting periods.

Where to power laws come from?

Inspired by Nate Bradley (STRESS 2007):

Condition on particle size d:

$$P(X > x|d = r) = e^{-x/m}$$

$$m = kd$$

Average using grain size pdf:

$$P(X > x) = \int_0^\infty P(X > x|d = r)g(r)\,dr$$

Hill et al.[6] use inverse gamma:

$$g(r) = \frac{\beta^\alpha}{\Gamma(\alpha)} r^{-\alpha-1} \exp\left(-\frac{\beta}{r}\right)$$

This leads to a power law step length:

$$\begin{aligned} P(X > x) &= \int_0^\infty e^{-\frac{x}{kr}} \frac{\beta^\alpha}{\Gamma(\alpha)} r^{-\alpha-1} \exp\left(-\frac{\beta}{r}\right)\,dr \\ &= \int_0^\infty e^{-\frac{x}{k}y} \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} \exp\left(-\beta y\right)\,dy \\ &= \left(1 + \frac{x}{k\beta}\right)^{-\alpha} \end{aligned}$$

See also Ganti et al. [5] and Stark et al. [16].

Tempered power laws

Exponential tempering: $P(X > x) \approx Cx^{-\alpha} \rightarrow Cx^{-\alpha} e^{-\lambda x}$

Tempered ADE: $\partial_t p_\lambda(x,t) = c\,\partial_x^{\alpha,\lambda} p_\lambda(x,t)$

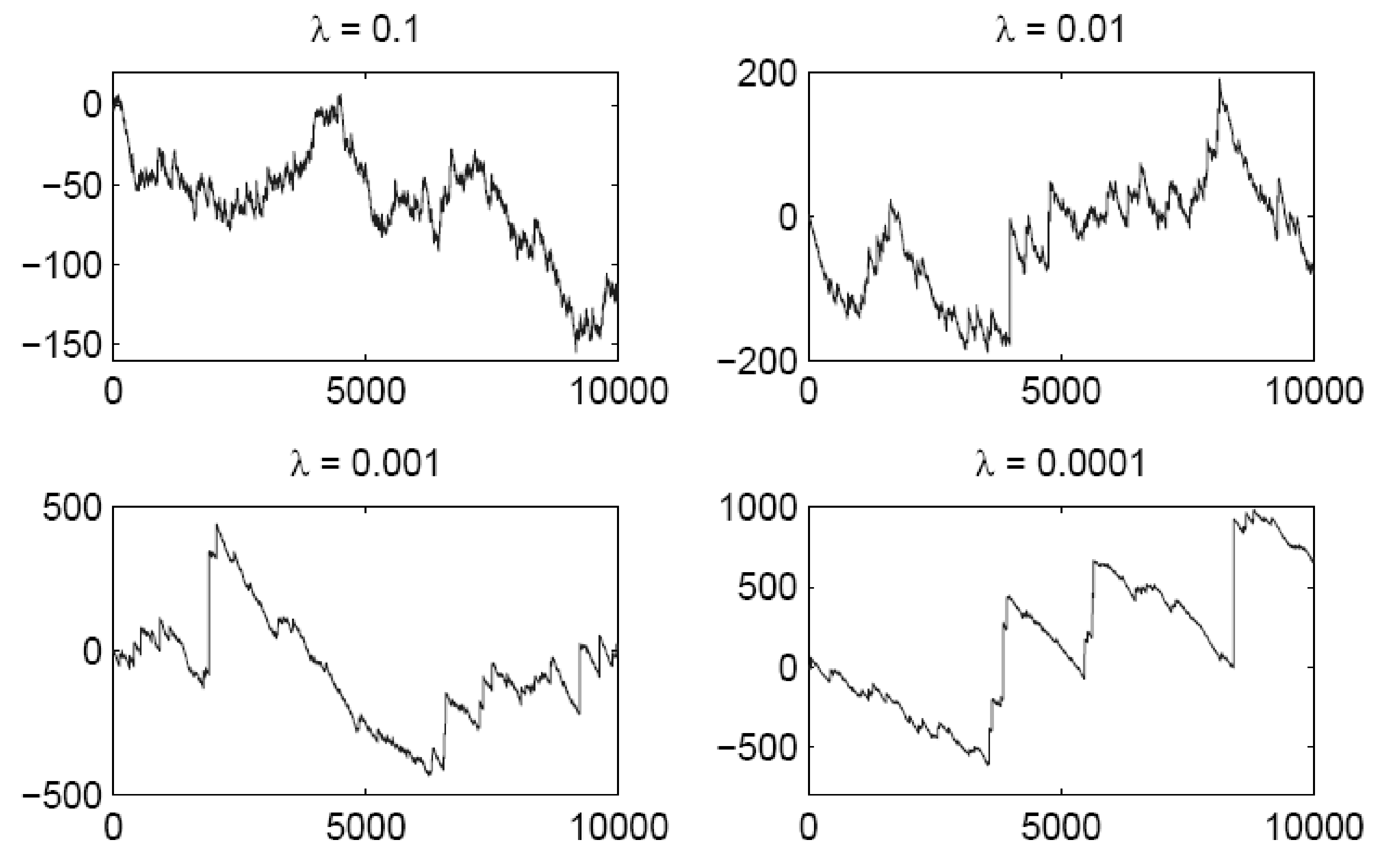
Tempered fractional derivative:

$$\partial_x^{\alpha,\lambda} f(x) = e^{-\lambda x}\,\partial_x^{\alpha}[e^{\lambda x} f(x)] - \lambda^{\alpha} f(x) - \alpha\lambda^{\alpha-1}\partial_x f(x)$$

Leads to a model that resembles fractional ADE at short time, classical ADE at long time.

Euler finite difference solutions, and particle tracking solutions, are outlined in Baeumer et al. [1].

Tempered Lévy motion

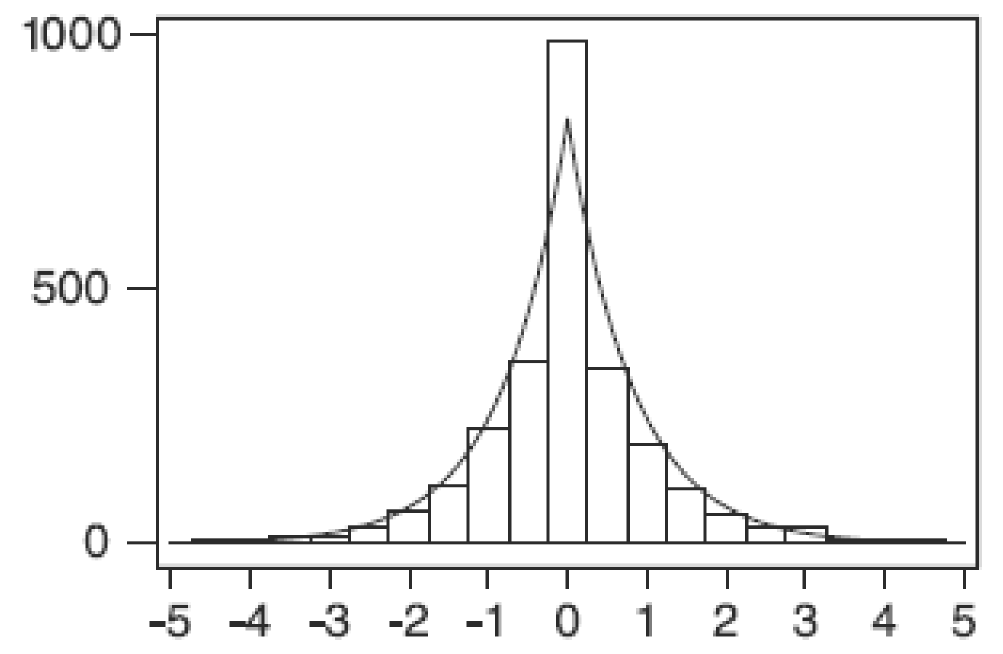


The tempering parameter λ cools the jumps.

Can also temper the time derivative [12].

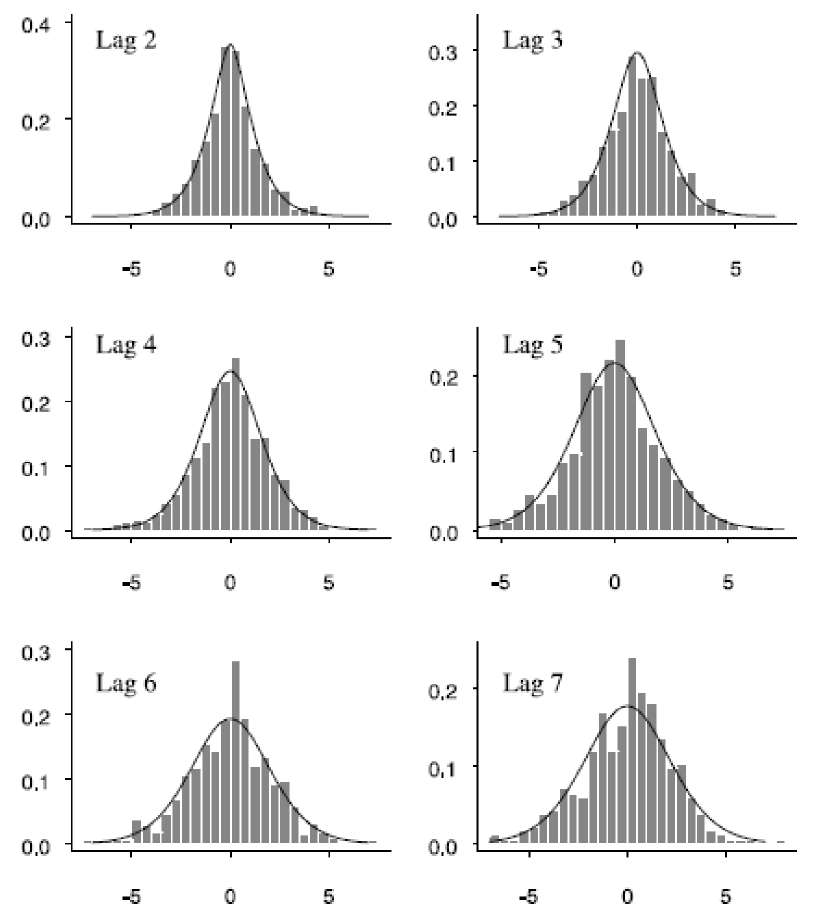
Fractional Laplace motion

Laplace pdf: $P(X > x) = \frac{\lambda}{2} e^{-\lambda|x|}$



Hydraulic conductivity increments at the MADE site [9].

Long time limit for Laplace motion



Sums of Laplace variables (here, the MADE data) are eventually normal.

Fractional Laplace motion

Fractional Brownian motion + exponential waiting times

$$\mathbb{E}X(t)X(s) = \frac{\sigma^2}{2}\left(\frac{\Gamma(2H+t/\nu)}{\Gamma(t/\nu)} + \frac{\Gamma(2H+s/\nu)}{\Gamma(s/\nu)} - \frac{\Gamma(2H+|s-t|/\nu)}{\Gamma(|s-t|/\nu)}\right)$$

Exponential cousin of the fractional derivative (here H=1/2):

$$\frac{\partial q(x,t)}{\partial t} = \frac{1}{\lambda}E_\lambda^+ q(x,t) + \frac{1}{\lambda}E_\lambda^- q(x,t),$$

$$E_\lambda^\pm f(x) = \int_0^\infty \frac{f(x-y) \mp f(x)}{y}\lambda e^{-\lambda y}\,dy.$$

Ganti et al. [4] apply to sediment transport in SAFL flume.

Multifractal properties are explored in Ganti et al. [4].

Vector fractional ADE

$$\frac{\partial C(x,t)}{\partial t} = -v \cdot \nabla C(x,t) + D\nabla_M^{\mathbf{E}} C(x,t)$$

The vector fractional derivative is a Cushman-type convolution [3]:

$$\boxed{\nabla_M^{\mathbf{E}} C(x,t) = \int \Big(C(x-y,t) - C(x,t) + y \cdot \nabla C(x,t) \Big)\ \phi(dy)}$$

$$\phi(dy) = \frac{m(\theta)}{r^2}\, dr\, d\theta \qquad\qquad y = r^{\mathbf{E}}\theta$$

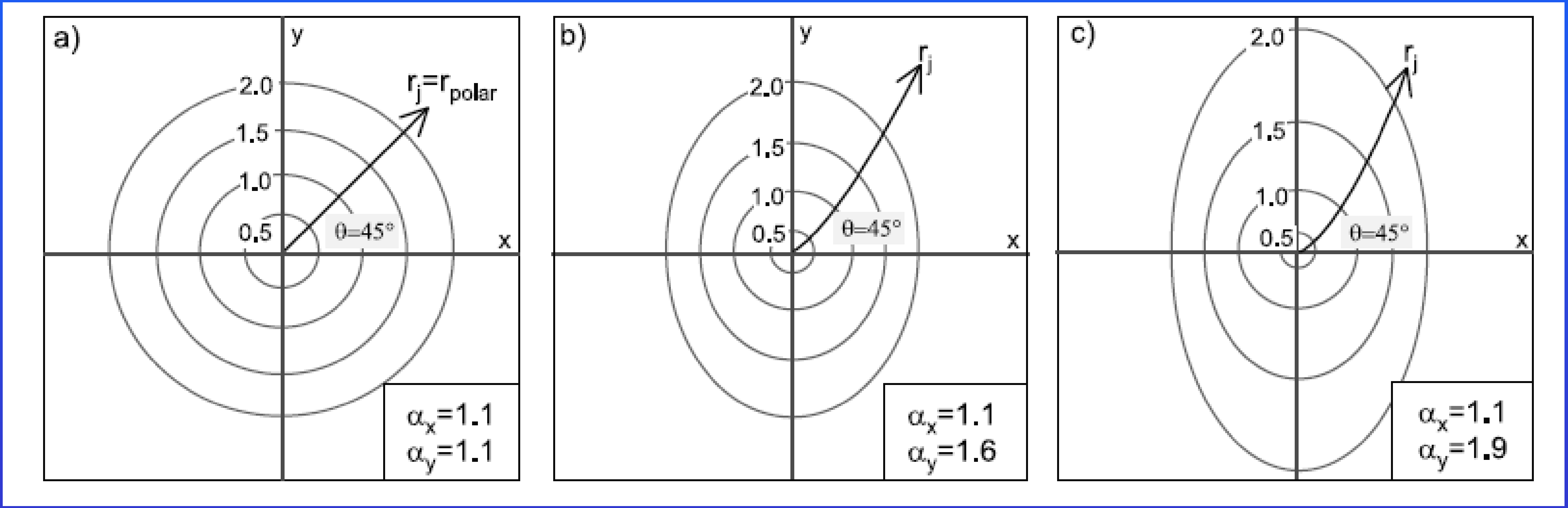
Eigenvalues of the matrix *E* determine power law jumps.

The mixing measure *m(θ)* determines jump directions.

Mixing measure

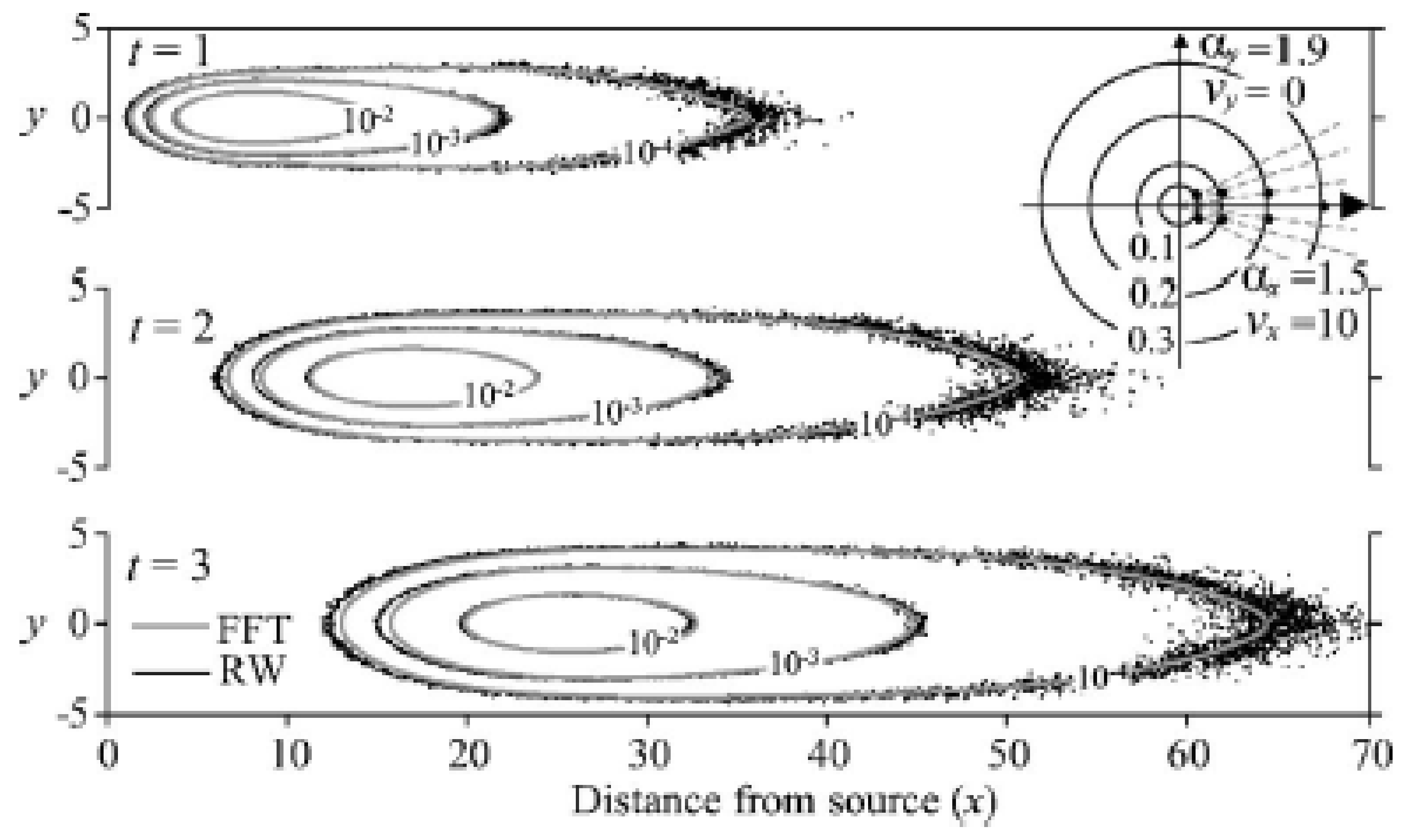
The mixing measure $m(\theta)d\theta$ codes the large jump directions.

Write vector jump in polar form: $(X,Y) = r^E\theta$



For more details see Schumer et al. [13].

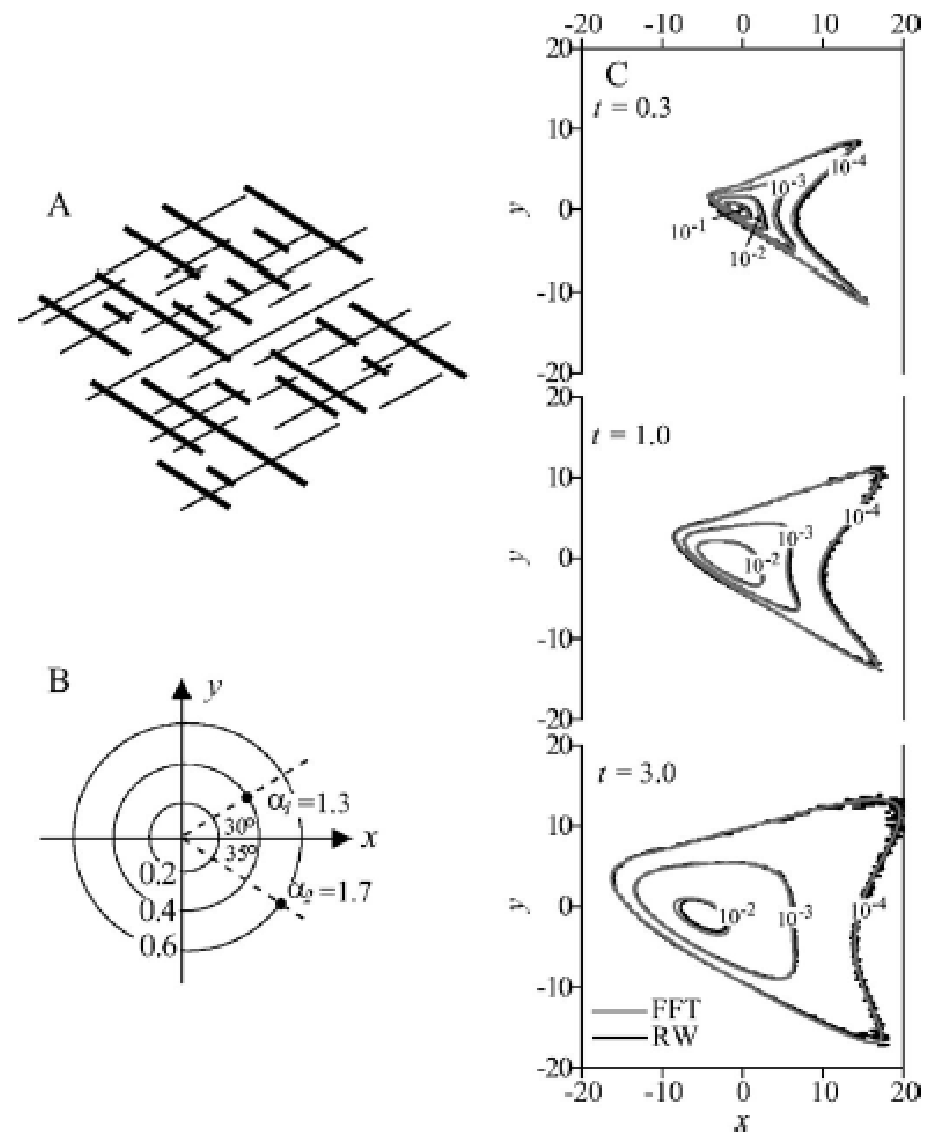
Vector fractional ADE for ground water flow



Discrete mixing measure, and different spreading rate in each coordinate.
Coordinates are determined by the flow direction

Particle tracking solutions by Zhang et al. [18].

The mixing measure M governs particle jump direction



Conceptual operator stable model for fracture flow from Schumer et al. [13].

Solutions via inverse FFT and particle tracking from Zhang et al. [18].

Finite difference methods are discussed in Tadjeran et al. [17] and [10].

Conclusions

- Fractional derivatives ↔ power laws
- Long jumps ↔ space-fractional ADE
- Long waiting times ↔ time-fractional ADE
- Need models and data to support power laws
- Power laws can be tempered
- Exponential alternatives to power laws
- Multifractal properties should be explored
- Vector models are a useful extension
- Lots and lots of open problems!

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